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DIMENSION SUBGROUPS AND SCHUR MULTIPLICATOR

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1. Introduction

Our main aim in this paper is to show that certain problems ([3], [11], [14]) about the integral dimension subgroups can be translated into equivalent problems about the Schur multiplicator $H^2(G,T)$, where T denotes the additive group of rationals mod 1 regarded as a trivial module over the group G. Let $\{P_nH^2(G,T)\}$ be the filtration of $H^2(G,T)$ as defined in [11, p. 64]. We prove that

(i) The integral dimension series of every nilpotent group terminates with identity in a finite number of steps if and only if, for every nilpotent group G, $P_nH^2(G,T)=H^2(G,T)$ for some $n\geq 1$.

(ii) The lower central series and the integral dimension series have equal intersections for every group if and only if, for every nilpotent p-group G without elements of infinite p-height, $\bigcup_n P_nH^2(G,T_p)=H^2(G,T_p)$, where T_p is the p-torsion subgroup of T. We also obtain some positive results about the filtration $\{P_nH^2(G,T)\}$. We prove that if G is a nilpotent group which is either finitely generated or torsion-free, then $P_nH^2(G,T)=H^2(G,T)$ for some $n\geq 1$. For arbitrary groups we show that there are constants $d_1,d_2,\dots,d_n,\dots$ such that $d_nH^2(G,T)\leq P_nH^2(G,T)$ for every nilpotent group G of class $\leq n$.

2. Integral dimension series and a filtration of Schur multiplicator

Let G be a group. We denote by $\Delta(G)$ the augmentation ideal of the integral group ring $\mathbb{Z}G$. Let $G=D_1(G)\geq D_2(G)\geq\cdots\geq D_n(G)\geq\cdots$ be the integral dimension series of G, $D_n(G)=G\cap(1+\Delta^n(G))$. There is a well-known correspondence (see [4, Chapter VI, §10]) between the elements of the cohomology group $H^2(G,M)$ and the equivalence classes of central extensions $1\to M\to\Pi\to G\to 1$ of M by G, when M is a trivial G-module. If M is divisible abelian, then we can easily identify by [10, Theorem 2.1] the central extensions corresponding to the elements of $P_nH^2(G,M)$.

2.1. Proposition. *Let G be a group, A a divisible abelian group regarded as a trivial*

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G-module, $\xi \in H^2(G, A)$ and $1 \to A \to \Pi \to G \to 1$ a central extension corresponding to ξ. Then $\xi \in P_n H^2(G, A)$ if and only if

$$A \cap (1 + \Delta^{n+2}(\Pi) + \Delta(\Pi)\Delta(A)) = (1).$$

The term $\Delta(\Pi)\Delta(A)$ occurring in Proposition 2.1 can be dropped when Π is a torsion group. For, in this case, $\Delta(\Pi)\Delta(A) \le \Delta^n(\Pi)$ for all $n \ge 1$ (see [11, Theorem 2.3, p. 97]).

Thus we have

2.2. Proposition. *Let G be a torsion group, A a torsion divisible abelian group regarded as a trivial G-module, $\xi \in H^2(G, A)$ and $1 \to A \to \Pi \to G \to 1$ a central extension corresponding to ξ. Then $\xi \in P_n H^2(G, A)$ if and only if $A \cap D_{n+2}(\Pi) = (1)$.*

The following result which is again a consequence of [10, Theorem 2.1] is handy to apply.

2.3. Lemma. *Let Π be a group, A a central subgroup of Π, B a divisible abelian group such that for every element $1 \ne x \in A$ there exists a homomorphism $f: A \to B$ with $f(x) \ne 0$. Then*

(i) $\quad A \cap (1 + \Delta^{n+2}(\Pi) + \Delta(\Pi)\Delta(A)) = (1) \quad$ *if* $P_n H^2(\Pi/A, B) = H^2(\Pi/A, B)$

and

(ii) $\quad A \cap \left(1 + \bigcap_n (\Delta^n(\Pi) + \Delta(\Pi)\Delta(A))\right) = (1) \quad$ *if* $\bigcup_n P_n H^2(\Pi/A, B) = H^2(\Pi/A, B)$.

We denote the centre of a group G by $\zeta(G)$ and set

$$I_n(G) = G \cap (1 + \Delta^n(G) + \Delta(G)\Delta(\zeta(G))) \quad \text{for } n \ge 1.$$

2.4. Lemma. *Every torsion nilpotent group G can be embedded in a nilpotent group G^* such that*

(i) *$I_n(G) \le D_n(G^*)$ for all $n \ge 1$, and*

(ii) *class of G = class of G^*.*

Proof. Let G be a torsion nilpotent group and A any divisible abelian group containing $\zeta(G)$. Form the central product

$$G^* = GA = G \times A / \{z^{-1}\phi(z) : z \in \zeta(G)\}$$

where $\phi: \zeta(G) \to A$ is a monomorphism. Then class of G = class of G^* and, since $\Delta(\zeta(G))\Delta(G) \le \Delta(A)\Delta(G) \le \Delta^n(G^*)$, we have $I_n(G) \le D_n(G^*)$.

2.5. Theorem. *The following statements are equivalent:*

(i) *For every nilpotent group G, there exists an integer $n \ge 1$ such that $D_n(G) = (1)$.*

(ii) *For every nilpotent group G, there exists an integer $n \geq 1$ such that $P_n H^2(G,T) = H^2(G,T)$.*

Proof. If (i) holds, then there exists a function $f(c)$ such that $D_{f(c)}(G) = (1)$ for every nilpotent group of class $\leq c$. By Lemma 2.4 and standard reductions we must also have $I_{f(c)}(G) = (1)$ for all such groups.

Let G be a nilpotent group of class c. If $\xi \in H^2(G,T)$ and $1 \to T \to \Pi \to G \to 1$ is a corresponding central extension, then Π is nilpotent of class $\leq c+1$. Therefore, $I_{f(c+1)}(\Pi) = (1)$ and, by Proposition 2.1, we have $P_{f(c+1)-2} H^2(G,T) = H^2(G,T)$.

The converse follows from Lemma 2.3 and induction on the class of G.

Let $\gamma_n(G)$ denote the nth term in the lower central series of a group G. Write

$$\gamma_\omega(G) = \bigcap_r \gamma_n(G) \quad \text{and} \quad D_\omega(G) = \bigcap_n D_n(G).$$

2.6. Theorem. *The following statements are equivalent:*

(i) *For every group G, $D_\omega(G) = \gamma_\omega(G)$.*

(ii) *For every nilpotent p-group G without elements of infinite p-height, $\bigcup_n P_n H^2(G, T_p) = H^2(G, T_p)$.*

Proof. Suppose (i) holds. Let G be a nilpotent p-group, $\xi \in H^2(G, T_p)$ and $1 \to T_p \to \Pi \to G \to 1$ a central extension corresponding to ξ. Then Π is a nilpotent p-group and $D_\omega(\Pi) = (1)$. Since T_p satisfies the minimum condition on subgroups, it follows that $T_p \cap D_{n+1}(\Pi) = (1)$ for some $n \geq 1$. Therefore, by Proposition 2.2, $\xi \in P_{n-1} H^2(G, T_p)$ and so (ii) holds.

Conversely suppose (ii) holds. Let G be a nilpotent p-group. Let $G(p)$ be the subgroup consisting of elements of infinite p-height in G. Then $G(p) \leq \zeta(G)$ ([2], [6]) and $D_\omega(G) \leq G(p)$ (see [11, Proposition 1.3, p. 95]). As $G/G(p)$ is a nilpotent p-group without elements of infinite p-height,

$$\bigcup_n P_n H^2(G/G(p), T_p) = H^2(G/G(p), T_p).$$

Lemma 2.3, therefore, implies that

$$G(p) \cap \left(1 + \bigcap_n (\Delta^n(G) + \Delta(G)\Delta(G(p))\right) = (1).$$

Hence $D_\omega(G) = (1)$ for every nilpotent p-group G and we have (i) by [3, Corollary A2].

Let $I_\omega(G) = \bigcap_n I_n(G)$. We can in fact easily prove the following (we omit the details):

2.7. Theorem. *The following statements are equivalent:*

(i) *For every group G, $I_\omega(G) = \gamma_\omega(G)$.*

(ii) *For every nilpotent group G and prime p*

$$\bigcup_n P_n H^2(G, T_p) = H^2(G, T_p).$$

We now give some positive results about the filtration $\{P_n H^2(G, T)\}$.

2.8. Theorem. *Let G be a nilpotent group which is either finitely generated or torsion-free. Then there exists an integer $n \geq 1$ such that*

$$P_n H^2(G, T) = H^2(G, T).$$

Proof. *Case* I: *G finitely generated.* Let $1 \to R \to F \to G \to 1$ be a free presentation of G with F free of finite rank. Let $[F, R]$ be the subgroup generated by all elements of the type $f^{-1}r^{-1}fr$, where $f \in F$, $r \in R$. Write $\bar{F} = F/[F, R]$ and $\bar{R} = R/[F, R]$. Since $\Delta(\bar{F})$ is a polycentral ideal of $\mathbb{Z}\bar{F}$, it satisfies the weak Artin–Rees property (see [13, Chapter XI, Theorem 2.8]). Therefore, there exists an n such that

$$\Delta^{n+2}(\bar{F}) \cap \Delta(\bar{R})\mathbb{Z}\bar{F} \leq \Delta(\bar{R})\Delta(\bar{F}).$$

For this n it follows that

$$\bar{R} \cap (1 + \Delta^{n+2}(\bar{F}) + \Delta(\bar{R})\Delta(\bar{F})) = (1).$$

Hence, by [12, Corollary 3.2],

$$P_n H^2(G, T) = H^2(G, T).$$

Case II: *G torsion-free.* Let G be a torsion-free nilpotent group of class c. Let $1 \to T \to \Pi \to G \to 1$ be a central extension. It can be deduced easily from [3, Lemma 4.1] that

$$\Delta^m(\Pi) \cap \mathbb{Z}(\Pi)\Delta(T) \leq \Delta(\Pi)\Delta(T) \quad \text{where } m = c^2 + 4c + 3.$$

Therefore,

$$T \cap (1 + \Delta^m(\Pi) + \Delta(\Pi)\Delta(T)) = (1)$$

and it follows from Proposition 2.1 that

$$P_{m-2} H^2(G, T) = H^2(G, T).$$

Every nilpotent p-group G has the property that $D_i(G) = \gamma_i(G)$ for $1 \leq i \leq p+1$ ([17], see also [7]). From this result and Proposition 2.2 we can deduce

2.9. Theorem. *If G is a nilpotent p-group of class $c < p$, then $P_c H^2(G, T) = H^2(G, T)$.*

With the help of [3, Lemma 6.2] we can prove

2.10. Theorem. *Let G be a torsion nilpotent group, K a normal subgroup of G supplemented by a finite p-subgroup and suppose that*

$$\bigcup_n P_n H^2(K, T_q) = H^2(K, T_q)$$

for every prime q. Then

$$\bigcup_n P_n H^2(G, T_q) = H^2(G, T_p) \quad \textit{for every prime } q.$$

In view of [9, Theorem 3.6] we have

2.11. Corollary. *Let G be a nilpotent p-group having an abelian normal subgroup of finite index. Then*

$$\bigcup_n P_n H^2(G, T_q) = H^2(G, T_q) \quad \textit{for every prime } q.$$

Sjogren [17] has given constants $c_1, c_2, \dots, c_n, \dots$ such that, for every group G, $D_n^{c_n}(G) \le \gamma_n(G)$ for all $n \ge 1$. We find that this result is equivalent to an analogous property of the filtration $\{P_n H^2(G, T)\}$.

2.12. Theorem. *The following statements are equivalent:*

(i) *There exist constants $c_1, c_2, \dots, c_n, \dots$ such that for every group G, $D_n^{c_n}(G) \le \gamma_n(G)$ for all $n \ge 1$.*

(ii) *There exist constants $d_1, d_2, \dots, d_n, \dots$ such that, for every nilpotent group G of class $\le n$, $d_n H^2(G, T) \le P_n H^2(G, T)$.*

Proof. (i)⇒(ii). Using Lemma 2.4 it can be shown that if (i) holds, then, for every group G, $I_n(G)^{c_n} \le \gamma_n(G)$ for all $n \ge 1$.

Let G be a nilpotent group of class $\le n$, $\xi \in H^2(G, T)$ and $1 \to T \to \Pi \to G \to 1$ a central extension corresponding to ξ. Let $\theta : T \to T$ be the homomorphism $\theta(t) = c_{n+2} t$. Since Π is a nilpotent group of class $\le n+1$, θ vanishes on $T \cap I_{n+2}(\Pi)$ and therefore on $T \cap (1 + \Delta^{n+2}(\Pi) + \Delta(\Pi)\Delta(T))$. Let

$$\alpha : T \to \mathbb{Z}\Pi / \Delta^{n+2}(\Pi) + \Delta(\Pi)\Delta(T)$$

be the homomorphism $t \mapsto t - 1 + \Delta^{n+2}(\Pi) + \Delta(\Pi)\Delta(T)$. Since T is divisible abelian and θ vanishes on $\operatorname{Ker} \alpha$, there exists a homomorphism

$$\phi : \mathbb{Z}\Pi / \Delta^{n+2}(\Pi) + \Delta(\Pi)\Delta(T) \to T$$

such that $\theta = \varphi \circ \alpha$. Let $\theta^* : H^2(G, T) \to H^2(G, T)$ be the homomorphism induced by θ. It follows from [10, Theorem 2.1] that $\theta^*(\xi) \in P_n H^2(G, T)$. However, $\theta^*(\xi) = c_{n+2}\xi$. Hence $c_{n+2} H^2(G, T) \le P_n H^2(G, T)$. We may thus take $d_n = c_{n+2}$ for all $n \ge 1$.

(ii)⇒(i). Suppose there exist constants $d_1, d_2, \dots, d_n, \dots$ as in (ii). We assert that the constants $c_1, c_2, \ldots, c_n, \ldots$ defined inductively by

$$c_1 = 1, \qquad c_2 = 1, \qquad c_{n+1} = c_n d_{n-1} \quad \text{for } n \geq 2$$

have the property that, for every group G, $I_n^{c_n}(G) \leq \gamma_n(G)$ for all $n \geq 1$. It is not hard to see that it is enough to establish the assertion for prime power groups only (see [8, Section 3]). We proceed by induction on n. The assertion holds trivially for $n = 1, 2$. Suppose it holds for all m with $1 \leq m \leq n$. Let G be a prime power group. To show that

$$I_{n+1}^{c_{n+1}}(G) \leq \gamma_{n+1}(G),$$

we may assume that $\gamma_{n+1}(G) = (1)$. Let $x \in I_{n+1}(G)$. Then, by induction $x^{c_n} \in \gamma_n(G) \leq \zeta(G)$. If $x^{c_n d_{n-1}} \neq 1$, then we can define a homomorphism $\beta : \zeta(G) \to T$ such that $\beta(x^{c_n d_{n-1}}) \neq 0$. Let $\alpha = d_{n-1}\beta$ and let α^*, β^* be the homomorphisms

$$H^2(G/\zeta(G), \zeta(G)) \to H^2(G/\zeta(G), T)$$

induced by α, β respectively. Let ξ be the element of $H^2(G/\zeta(G), \zeta(G))$ which corresponds to the central extension

$$1 \to \zeta(G) \to G \to G/\zeta(G) \to 1.$$

Then $\alpha^*(\xi) = d_{n-1}\beta^*(\xi) \in d_{n-1}H^2(G/\zeta(G), T) \leq P_{n-1}H^2(G/\zeta(G), T)$. Therefore, by [10, Theorem 2.1] α can be extended to a map $\phi : G \to T$ whose linear extension to $\mathbb{Z}G$ vanishes on $\Delta^{n+1}(G) + \Delta(G)\Delta(\zeta(G))$. But then

$$\beta(x^{c_n d_{n-1}}) = \alpha(x^{c_n}) = \phi(x^{c_n} - 1) = 0,$$

a contradiction. Hence we must have $x^{c_n d_{n-1}} = 1$ and the induction is complete.

2.13. Corollary. *For the constants $c_1, c_2, \ldots, c_n, \ldots$ defined by Sjogren,*

$$c_{n+2}H^2(G, T) \leq P_n H^2(G, T)$$

for every nilpotent group G of class $\leq n$.

2.14. Remarks. (i) Let G be a p-group of class 3. If $p \neq 2$, then the integral dimension series and the lower central series of G coincide [8]. On the other hand, there are 2-groups of class 3 with $D_4(G) \neq (1)$ ([15], [18], [19]). For further insight into dimension subgroups it will, therefore, be of interest to first know whether the dimension series of 2-groups of class 3 have bounded lengths.

(ii) For positive results about the statements in Theorems 2.5 and 2.6 we refer the reader to [1], [2], [3], [5], [14], [16], [20] and [21].

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